Non-Maximal Rank Separable States Are A Set Of Measure Zero Within The Set of Non-Maximal Rank States

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Abstract

It is well known that the set of separable pure states is measure 0 in the set of pure states. Herein we extend this fact and show that the set of rank r separable states is measure 0 in the set of rank r states provided r is not maximal rank.

Recently quite a few authors have looked at low rank separable and entangled states. (See [1] and the references therein and [2].) Therefore it makes sense to determine the size of the set of rank r separable states within the set of rank r states. For rank 1, it is well known that the separable states are a set of measure zero. This contrasts with the maximal rank case, where the separable states not only are not measure zero, but contain an open set.

The purpose of this note is to show that the maximal rank case is the exception. In particular, suppose we have p particles modelled on the Hilbert space $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p}$. Then the following is true.

Theorem 1 *Let S_r be the set of rank r separable matrices on $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p}$ and D_r the set of all rank r density matrices. S_r is measure 0 in D_r, for all $r < N = n_1 \cdots n_p$.*

The proof will use Sard's Theorem [3]to show the set of ranges of separable rank r density matrices is measure 0 in the set of ranges of rank r density matrices, i.e. within the set of r-dimensional subspaces of $\mathbb{C}^N$; i.e. within $G(N,r)$, the Grassmann manifold of $r-$planes in $\mathbb{C}^N$ [4]. That this is what we need to consider follows from the fact D_r is $G(N,r) \times Herm_1^+(r)$, where $Herm_1^+(r)$ is the space of Hermitian $r \times r$ matrices that are positive definite and trace 1. For those not familiar with Sard's Theorem, it is an extension to nonlinear functions of the well known fact that if T is a linear transformation between two finite dimensional vector spaces, V and W, and the rank of T is less than the dimension of W, then $T(V)$ has measure zero in W, since it is a lower dimensional space. In the nonlinear case with which Sard's theorem deals, the image of $f : M \to N$ has measure 0 if at each point of M the derivative (Jacobian matrix) of f has rank less than the dimension of N.

Lemma 2 *Suppose A and B are positive semi-definite linear operators. Then $Ker(A+B) = KerA \cap KerB$ and $Range(A+B) = RangeA + RangeB$.*

Proof. Clearly $KerA \cap KerB \subset Ker(A+B)$ and $Range(A+B) \subset RangeA + RangeB$. Suppose $v \in KerA \cap KerB$. Then $0 = \langle (A+B)v, v\rangle = \langle Av, v\rangle + \langle Bv, v\rangle$. Since A and B are positive semi-definite, this means $Av = 0$ and $Bv = 0$, hence $KerA \cap KerB = Ker(A+B)$. Since $Ker(A+B)$ is the orthogonal complement of $Range(A+B)$, it follows that $KerA \cap KerB$ is the orthogonal complement of $Range(A+B)$. But $KerA \cap KerB$ is the orthogonal complement of $Range(A+B)$, so $Range(A+B) = RangeA + RangeB$. ■

If A is a separable density matrix, then A is the convex combination of projections onto product states. It follows from the lemma that the range of A therefore has a basis of product states. The product states are precisely the image of $\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1}$in $\mathbb{P}^{N-1}$ under the map induced by tensor product on $\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_p}$, where $\mathbb{P}^k$ is k-dimensional complex projective space (i.e. the space of rays in $\mathbb{C}^{k+1}$).

As for the manifold of r-dimensional subspaces of $\mathbb{C}^N$, i.e., the Grassmann manifold, $G(N,r)$, it is obtained by first considering $\mathbb{C}^{Nr}$ as being the set of $N \times r$ complex matrices and taking $L(N,r)$ to be the open subset consisting of those matrices with rank r. $G(N,r)$ is then the orbit space $GL(r,\mathbb{C})\backslash L(N,r)$.

Let $\widetilde{\mu} : \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_p} \to \mathbb{C}^N$ be tensor product and take $\mu : (\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_p})^r \to \mathbb{C}^{Nr}$ to be $(\widetilde{\mu}, ..., \widetilde{\mu})$. Let $Q = \mu^{-1}(L(N,r))$. Thus the image of Q under μ is the set of $N \times r$ matrices with columns that are product vectors. And if π:$L(N,r) \to G(N,r)$ is the projection, then $\pi \circ \mu : Q \to G(N,r)$ has as its range the $r-$dimensional subspaces that have a basis of product states. The proof of the theorem is that the derivative of $\pi \circ \mu$ always has rank less than $\dim(G(N,r))$.

To see this, first note that due to the homogeneity of the spaces involved the rank of $d\pi \circ \mu$ is the same at each point. Therefore we need to compute it at only one point. The point we choose is one whose image in $L(N,r)$ under μ is P, where P is the matrix whose first r rows form the $r \times r$ identity and whose last $(N-r)$ rows are zero. If for $T \in L(N,r)$, we take T_1 to be the $r \times r$ matrix consisting of the first r rows of T and T_2 to be the $(N-r) \times r$ matrix consisting of the last $N-r$ rows of T, then $\mathcal{U} = \{T : T_1 \text{ is invertible}\}$ is an open neighborhood of P. Furthermore on $\mathcal{U}$, π can be taken to be $\pi(T) = T_2 T_1^{-1}$. In which case, $d\mu(T)(S) = (S_2 - T_2 T_1^{-1} S_1) T_1^{-1}$ and so $d\mu(P)(S) = S_2$.

Now, one point $\mathbf{v} = (v_1, ...v_r) \in Q$ which μ maps to P has

$$v_1 = ((1, 0..., 0), ..., (1, 0, ..., 0)). \tag{1}$$

If $\mathbf{h} = ((h_{11}, ..., h_{1n_1}), ..., (h_{p1}, ..., h_{pn_p}))$, then $d\mu(\mathbf{v})(\mathbf{h})$ is the linear part in h_{ij}

of

$$(1 + h_{11}, h_{12}, ..., h_{1n_1}) \otimes \cdots \otimes (1 + h_{p1}, h_{p2}, ..., h_{pn_p}) \qquad (2)$$

It is clear that the last entry in this linear part is 0. Therefore if $S = d\mu(\mathbf{v})(\mathbf{h})$, then S always has a zero in the last entry in the first column. Hence the range of $d\mu(\mathbf{v})$, is less than $(N-r)r = \dim(G(N,r)$.

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